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Numerical study of local and global persistence in directed percolation

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Abstract. The local persistence probability $P_l(t)$ that a site never becomes active up to time t, and the global persistence probability $P_g(t)$ that the deviation of the global density from its mean value $\rho(t)-\langle\rho(t)\rangle$ does not change its sign up to time t are studied in a (1+1)-dimensional directed percolation process by Monte-Carlo simulations. At criticality, starting from random initial conditions, $P_l(t)$ decays algebraically with the exponent $\theta_l \approx 1.50(2)$. The value is found to be independent of the initial density and the microscopic details of the dynamics, suggesting θ_l is an universal exponent. The global persistence exponent θ_g is found to be equal or larger than θ_l. This contrasts with previously known cases where $\theta_g < \theta_l$. It is shown that in the special case of directed-bond percolation, $P_l(t)$ can be related to a certain return probability of a directed percolation process with an active source (wet wall).

PACS. 64.60.Ak Renormalization-group, fractal and percolation studied of phase transition – 05.40.+j Fluctuation phenomena, random processes, and Brownian motion – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

1 Introduction

In recent years it has been realized that certain first-passage quantities in nonequilibrium systems exhibit a power law decay with non-trivial exponents [1–12]. One of these quantities is the local persistence probability $P_l(t)$, defined as the probability that a local variable at a given point in space (normally a spin) has not changed its state until time t during a stochastic evolution. In various systems it was found that $P_l(t) \sim t^{-\theta_l}$, where θ_l is the *local persistence exponent*. A similar quantity, the global persistence probability $P_g(t)$, defined as the probability that the global order parameter does not change its sign up to time t, is also found to decay as a power law with a *global persistence exponent* θ_g. In general the exponents θ_l and θ_g are found to be independent of the usual scaling exponents and different from each other. The persistence probabilities depend on the history of evolution as a whole, and thus it is generally hard to determine these exponents analytically. Only a few cases of exact results are known [2,4,11].

An important nonequilibrium process that has been studied extensively is directed percolation (DP) [13], interpreted as a reaction-diffusion system. A large variety of physical systems which undergo a phase transition from a fluctuating active phase into an absorbing state (*i.e.* a configuration once reached, the system cannot escape from) belongs to the same universality class as DP. Therefore it would be interesting to study the persistence probability in DP since this could provide some understanding of the non-equilibrium nature of this process.

In this article we present a numerical study of the local and the global persistence probabilities in (1+1)-dimensional DP processes. We find that the local persistence probability decays as a power law with the exponent $\theta_l \approx 1.5$. On the other hand, our measurements of the global persistence probability indicate that $\theta_g \geq 1.5$. This observation comes as a surprise since in all previously known cases it was found that $\theta_g < \theta_l$ [7]. Our findings are supported by a recent field-theoretical analysis of global persistence [14].

Persistence exponents are known to exhibit universal properties. For example, the local persistence exponent of the (2+1)-dimensional Glauber model in the ordered phase $T < T_c$ is independent of the temperature T[15], whereas it is not universal with respect to the initial magnetization [4]. In contrast, the persistence exponents in DP turn out to be independent of the initial density. In addition we find these exponents to be the same for different microscopic realizations of the DP process, indicating that θ_l and θ_g are indeed universal exponents that characterize the DP universality class.

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We also show that the local persistence probability is related to certain observables measured in (1+1)-dimensional DP processes with boundaries. Introducing a transfer matrix formalism we establish an exact relation between $P_l(t)$ and a specific quantity measured in a DP process with an absorbing boundary (dry wall). A similar relation is found between $P_l(t)$ (which can be seen as a "first return" probability) and the probability $R(t)$ that a DP process with a steady active source (wet wall) returns to its initial condition. For a particular realization of DP, called bond-directed percolation, this relation can be proven exactly. Thus the problem of local persistence is related to both the dry and the wet wall problems in DP which have been discussed in references [16–18].

The article is organized as follows: in Section 2 we briefly review the DP process. Our numerical results are presented in Section 3. Performing Monte-Carlo (MC) simulations we measure local and global persistence probabilities and estimate the corresponding exponents θ_l and θ_g. In order to measure θ_g more accurately we use a block spin method which has been introduced recently in reference [19]. In Section 4 the relation between $P_l(t)$ and a specific observable in a DP process with a *dry* wall is proved exactly. The relation between $P_l(t)$ and the return probability $R(t)$ in presence of an active source is discussed in Section 5.

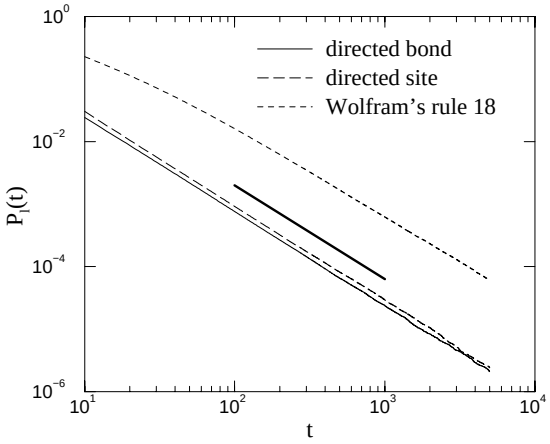

Fig. 1. Universal properties of the local persistence exponent θ_l. The probability $P_l(t)$ that a site never becomes active up to time t is measured in a Domany Kinzel model with 1024 sites averaged over 10^5 independent runs. Starting with the initial density $\rho(0) = 0.8$ various critical points are examined. The results indicate a power law decay which is consistent with $\theta_l = 1.49(2)$ in all cases. The bold line marks slope $-3/2$.

2 Directed percolation – a brief overview

Directed percolation [13] is used as a model for the spreading of some generally non-conserved agent and plays a role in certain autocatalytic chemical reactions and spreading of epidemics. In DP models, sites of a lattice are either occupied by a particle (active, wet) or empty (inactive, dry). The dynamical processes are defined in a way that particles can either self-destruct or produce an offspring at a neighboring empty site. If the rate for offspring production p is very low, the system will always reach a state without particles which is the absorbing state of the system. On the other hand, when p exceeds a critical threshold p_c, another steady state of the system exists on the infinite lattice, where the density $\rho(p)$ of active sites is finite. Between the two phases a continuous phase transition takes place which is characterized by long range correlations. There are various different models for DP, *e.g.* directed site and bond percolation on a lattice [13], cellular automata such as the Domany-Kinzel model [20], and the contact process [21], to name only a few.

One of the most important properties of directed percolation is its robustness with respect to the microscopic dynamics of the system. According to a widely accepted conjecture formulated by Janssen and Grassberger [22], any transition from a fluctuating active phase into a single, non-fluctuating and non-degenerate absorbing state belongs to the DP universality class, provided the dynamical processes are local and characterized by a one-component order parameter without special attributes like additional symmetries or frozen randomness. The DP universality class is characterized by three critical exponents, namely the density exponent β and the scaling exponents $\nu_{||}$ and $\nu_\perp$. Therefore, if $P_l(t)$ and $P_g(t)$ actually decay as a power law, an interesting question would be whether θ_l and θ_g are independent of these three exponents and exhibit a similar robustness.

In the present work we analyze the persistence probabilities $P_l(t)$ and $P_g(t)$ by Monte-Carlo simulations. A DP model which is convenient for this purpose is the Domany-Kinzel (DK) cellular automaton [20]. The DK model is defined as follows: a binary variable $\sigma_i(t) = 0, 1$ characterizes the state of site i at discrete time t. $\sigma = 1$ means that the site is active (wet) whereas $\sigma = 0$ means that it is inactive (dry). The automaton evolves by a parallel update rule in which the state of $\sigma_i(t+1)$ is selected according to transition probabilities $\tau(\sigma_i(t+1)|\sigma_{i-1}(t), \sigma_{i+1}(t))$. The transition probabilities of the DK model are

$$\begin{aligned} \tau(1|0,0) &= 0 \\ \tau(1|0,1) = \tau(1|1,0) &= p_1 \\ \tau(1|1,1) &= p_2 \\ \tau(0|\sigma_{i-1},\sigma_{i+1}) &= 1 - \tau(1|\sigma_{i-1},\sigma_{i+1}). \end{aligned} \tag{1}$$

Thus the model is controlled by two parameters p_1 and p_2. For fixed $p_2 < 1$ there is a critical value $p_{1,c}$ where a continuous phase transition takes place. It is assumed that for $p_2 < 1$ the critical behavior along the whole line $p_{1,c}(p_2)$ is the same as that of DP. In particular, the phase transition line includes three particular points, namely [23]:

1. directed bond percolation: $p_1 = 0.644701(1)$, $p_2 = 2p_1 - p_1^2$;
2. directed site percolation: $p_1 = p_2 = 0.705485(5)$;
3. Wolfram's rule 18: $p_1 = 0.8092(1)$, $p_2 = 0$.

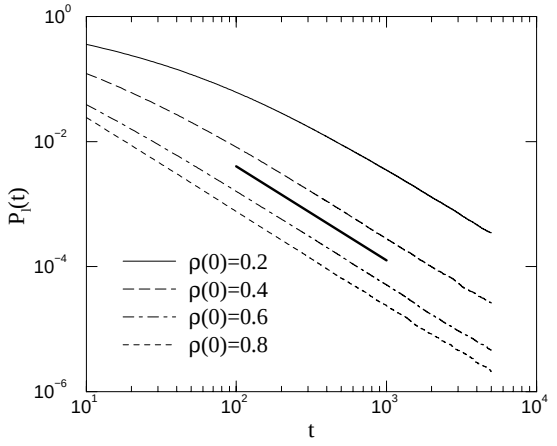

Fig. 2. Universal properties of the local persistence exponent θ_l. $P_l(t)$ is measured for directed bond percolation with various initial densities. For $\rho(0) \geq 0.4$ an algebraic decay with $\theta_l = 1.50(2)$ is observed. In the case of very low initial density there seems to be a longer transient until the final power law decay is reached. The bold line indicates the slope $-3/2$.

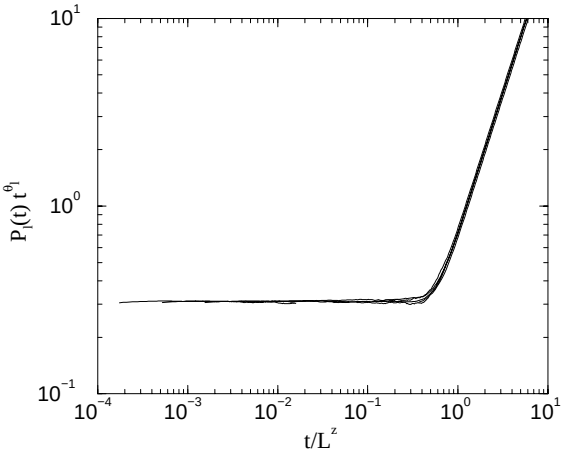

Fig. 3. Finite-size scaling function for the local persistence probability in the absorbing phase. The scaled persistence probability $P_l(t)t^{\theta_l}$ is measured at criticality for system sizes $L = 8, 16, 32, \ldots, 1024$ and agains scaled time t/L^z. The best data collapse is obtained for $\theta_l = 1.50(1)$.

As a necessary condition, a universal critical exponent should not depend on the choice of the transition point used in a simulation at criticality.

Another special case is *compact* directed percolation where $p_1 = 1/2$ and $p_2 = 1$. Here the dynamics of the model belong to the universality class of annihilating random walks or, equivalently, the Glauber Ising model. In fact, it turns out that in the case of compact DP the results of references [1,4,7] are recovered.

3 Monte-Carlo simulations

3.1 Local persistence probability

The local persistence probability $P_l(t)$ is defined as the probability that in a DP process starting from random initial conditions a given inactive site does not become active up to time t. This quantity is nontrivial because it can be seen as an infinite-point correlation function in the direction of time.

Notice that in contrast to Ising systems we define $P_l(t)$ as the probability for a site not to become active rather than not to flip. The reason is the asymmetry between active and inactive sites in DP: since active sites can spontaneously turn into inactive sites, the probability for a site to remain active up to time t decays exponentially. On the other hand, sites may remain inactive for very long times since they can only flip in presence of an active neighboring site. Thus $P_l(t)$, defined as the probability for a site to remain inactive, is expected to decay more slowly.

The qualitative behavior of $P_l(t)$ depends on the percolation probability. In the inactive phase the density of active sites decays exponentially fast until the system enters the absorbing state. Therefore, a finite fraction of sites remains inactive so that $P_l(t)$ saturates at some constant value. In the active phase an infinite percolating cluster emerges. Since in that case the size of inactive islands in the cluster is finite, $P_l(t)$ will decay exponentially. At the critical point, however, there is no characteristic length scale and therefore we expect $P_l(t)$ to decay as a power law.

We performed Monte-Carlo simulations at various transition points for different initial densities. In all cases a power law decay is observed. The local persistence exponent $\theta_l \approx 1.50(3)$ is found to be independent of the choice of the transition point as well as the initial density (see Figs. 1 and 2).

Similar results (not reported here) are obtained for the contact process [21] which is a model for DP with random sequential dynamics. Our findings therefore suggest that θ_l is indeed a universal exponent in DP.

In addition, we examined the scaling properties of the local persistence probability. According to the usual scaling theory of DP [13], we expect $P_l(t)$ to scale like

$$P_l(t, L, \epsilon) \sim t^{-\theta_l} \, f(\epsilon^{\nu_{||}} t, L^{-z} t) \,, \tag{2}$$

where $\epsilon = |p - p_c|$ measures the distance from the critical point, $z = \nu_{||}/\nu_\perp$ is the dynamical exponent, and f is a universal scaling function with the asymptotic behavior

$$\begin{aligned} f(0,0) &= const. \\ f(x,0) &\sim x^{\theta_l} \text{ for } \; x \to \infty \\ f(0,y) &\sim y^{\theta_l} \text{ for } \; y \to \infty \,. \end{aligned} \tag{3}$$

We verified the finite-size scaling of equation (2) at criticality. In Figure 3 the scaling function $f(0, L^{-z}t)$ is shown for various system sizes.

The best data collapse is obtained for $\theta = 1.50(1)$ which is our the most precise estimate for the local persistence exponent. Attempts to relate θ_l to the DP exponents $\beta \simeq 0.2765$, $\nu_\perp \simeq 1.0968$, and $\nu_{||} \simeq 1.7338$ [17,23] have failed wherefore we believe that θ_l is an independent critical exponent.

3.2 Global persistence probability

The global persistence probability $P_g(t)$ is usually defined as the probability that the global order parameter (*e.g.* the total magnetization in the Ising model) does not change its sign up to time t. It has been studied in various models as, for example, in the Ising model [7] and models in the parity conserving class [10]. In all cases it was found that $P_g(t)$ decays algebraically with an exponent θ_g which is independent of θ_l and the other scaling exponents of the system.

In DP, however, the global order parameter – the density of active sites $\rho(t)$ – is a strictly positive quantity and therefore the above definition is not applicable. Instead one has to consider the probability that the *deviation* of the order parameter from its mean value $\Delta\rho(t) = \rho(t) - \langle\rho(t)\rangle$ does not change its sign up to time t. As in the case of local persistence we observe that in *finite* systems this probability depends on the sign of $\Delta\rho(t)$. In the context of a recent field-theoretical analysis of global persistence in DP [14] it was predicted that this asymmetry should vanish in an infinite system where $\Delta\rho(t)$ becomes a Gaussian process. In finite systems, however, nonlinear contributions in the noise correlations play an important role. In fact, the normalized variance of the fluctuations is no longer constant but increases monotonically with the actual value of the deviation $\Delta\rho(t)$. This implies that the effective time scales for postive and negative fluctuations are different and therefore intermediate states with $\Delta\rho(t) > 0$ have a shorter lifetime than those with a negative deviation. In numerical simulations this asymmetry strongly affects the global persistence probability. Even in systems with several thousand sites the probability for the deviation to remain positive up to time t decays as a power law while the corresponding probability for negative deviation vanishes exponentially. In agreement with reference [14], we observe that this asymmetry gradually decreases with increasing system size, but as a matter of fact numerical simulations seem to be confined to a regime where the asymmetry dominates. Keeping this problem in mind, we define $P_g(t)$ as the probability of $\Delta_\rho(t)$ to remain *negative* from $t = 0$ up to time t. Notice that in contrast to reference [14], we do not introduce intial waiting times.

Usually it is difficult to measure θ_g by Monte-Carlo simulations, the reason being that each run produces only one data point (the first passage time) whereas the measurement of θ_l produces a number of data points of the order of the system size L. Hence a large number of runs is required to obtain good statistics. However, recently it has been shown [19] that this problem can be circumvented by introducing spin-block persistence probabilities

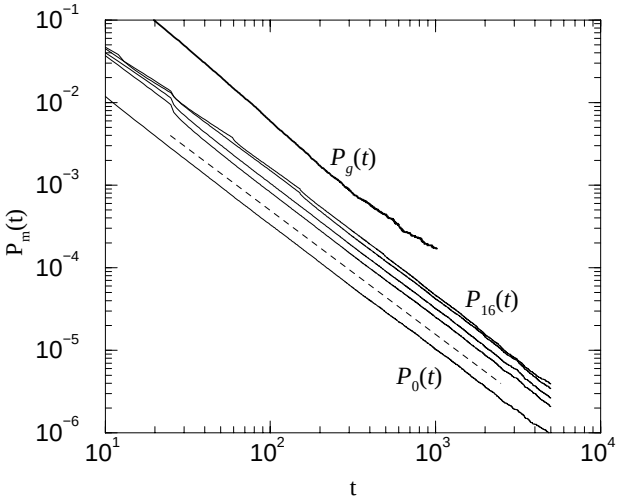

Fig. 4. Global persistence probability and Block spin method: Decay of the persistence probability $P_m(t)$ for block sizes $m = 1, 2, 4, 8, 16$. From the slopes we obtain the estimates $\theta_1 = 1.52(2)$, $\theta_2 = 1.48(2)$, $\theta_4 = 1.49(2)$, $\theta_8 = 1.52(2)$, $\theta_{16} = 1.49(2)$. The bold line represents a direct measurement of $P_g(t)$ (shifted vertically by a factor of 10). The dashed line indicates slope $-3/2$.

$P_m(t)$ which are defined as follows: consider blocks of m sites and define a block density $\rho_m(t)$ as the average fraction of active sites in each block. Let $P_m(t)$ be the probability that $\Delta\rho_m(t) = \rho_m(t) - \langle\rho(t)\rangle$ remains negative up to time t. Obviously the spin-block persistence probability $P_m(t)$ connects the special cases of local and global persistence, namely

$$P_1(t) = P_l(t), \qquad P_\infty(t) = P_g(t)\,. \tag{4}$$

It was observed in reference [19] that $P_m(t)$ in a Glauber model first decays as $P_g(t) \sim t^{-\theta_g}$ and then crosses over to the power law decay $P_m(t) \sim P_l(t) \sim t^{-\theta_l}$, where the crossover time grows with the box size. Since the number of such boxes is of order L/m, this method allows to measure θ_g much more accurately. We have measured the $P_m(t)$ for block sizes $m = 1, 2, 4, 8, 16$. As can be seen in Figure 4, $P_m(t)$ decays as $t^{-\theta_m}$ where the exponents $\theta_m \simeq 1.50(4)$ seem to be independent of m, suggesting that the global and the local persistence exponents are identical. However, the question arises whether the spin-block method is still valid in the case of DP since the small box sizes enhance the nonlinearity of the fluctuations mentioned above [24]. Already after 100 updates, the correlation length $\xi_\perp$ of a DP process extends over approximately 50 sites. Since the correlation length is larger than the considered box sizes it is questionable whether the spin-block method is applicable in DP. Even the direct measurement of global persistence remains problematic since ratio $L/\xi_\perp \simeq 20$ is probably so small that we are still in a regime where $\Delta_\rho(t)$ does not behave like a Gaussian process.

In order verify the spin-block results, we measured the global persistence probability $P_g(t)$ over two decades in time (see bold line in Fig. 4). Although this measurement is not very accurate, it indicates that the actual value θ_g might be equal or slightly larger than θ_l. This is surprising since in all previously known cases θ_g was found to be smaller than θ_l. For example, in the (1+1)-dimensional Glauber-Ising model $\theta_g = 1/4$ and $\theta_l = 3/8$ [4,7].

3.3 Power law versus stretched exponential

In certain reaction diffusion models it was shown that the $P_l(t)$ decays as a stretched exponential function rather than a power law [3]. In numerical simulations it is sometimes difficult to distinguish between stretched exponential and power law decay. In order to verify whether $P_l(t)$ truly decays as a power law one can use a heuristic argument as indirect test [3]: consider a site that has been inactive up to time t. In order to become active in the next time step $t+1$, it is necessary that a neighboring site is active at time t. Hence $P_l(t)$ changes according to

$$\frac{d}{dt}P_l(t) = -P_l(t)\rho_s(t)\,, \tag{5}$$

where $\rho_s(t)$ is the probability for finding an active site near a site which was never active until time t. By integrating equation (5) one can easily show that $P_l(t)$ decays as a power law only if $\rho_s(t) \sim t^{-1}$. On the other hand, if $\rho_s(t)$ decayed as $t^{-\alpha}$ with $\alpha \neq 1$, $P_l(t)$ would decay as a stretched exponential. Thus, by measuring the exponent α in a Monte-Carlo simulation, we can verify the observed power law decay for $P_l(t)$. Our results (see Fig. 5) are consistent with $\rho_s(t) \sim t^{-1}$, supporting that $P_l(t)$ actually decays algebraically.

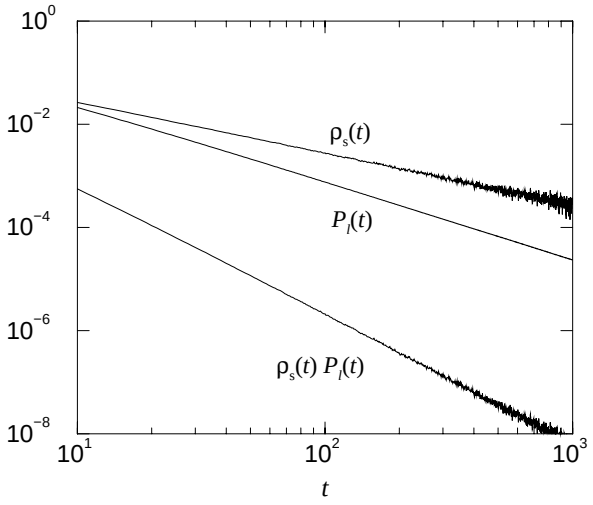

Fig. 5. Verification of the power law decay of $P_l(t)$. The probability $\rho_s(t)$ for finding an active site near a site which was never active until time t is plotted as a function of time. It is observed that $\rho_s(t) \sim t^{-\alpha}$ with $\alpha = 1.01(2)$, indicating that $P_l(t)$ decays indeed algebraically (see text).

4 Relation to directed percolation with an absorbing boundary

In this section we prove that the local persistence probability $P_l(t)$ is exactly equal to the expectation value of a specific observable in a DP process with an absorbing boundary. A boundary is called absorbing if all bonds across the boundary are cut. In a (1+1)-dimensional system such an absorbing boundary can be introduced by forcing a particular site to be inactive during the whole time evolution, *i.e.* an absorbing boundary can be understood as a dry wall. The effect of an absorbing boundary in a (1+1)-dimensional DP process has been recently studied in references [16–18], and it is therefore interesting to investigate the relation between the two problems.

Let us consider a DK model with L sites and periodic boundary conditions. Denoting by $|\sigma\rangle = \{\sigma_1, \sigma_2, \ldots, \sigma_L\}$ basis vectors in configuration space, the transfer matrix $\mathbf{T}$ of the DK model is defined by

$$\langle\sigma'(t+1)|\mathbf{T}|\sigma(t)\rangle = \prod_{i=1}^{L} \tau\Big(\sigma_i(t+1)|\sigma_{i-1}(t), \sigma_{i+1}(t)\Big)\,. \tag{6}$$

Furthermore let us define vectors for the absorbing configuration $|0\rangle$, the state $|a\rangle$ where all sites are active, and the sum over all configurations $|1\rangle$:

$$\begin{aligned} |0\rangle &= \{0, 0, \ldots, 0, 0\} \\ |a\rangle &= \{1, 1, \ldots, 1, 1\} \\ |1\rangle &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_L} |\sigma\rangle\cdot \end{aligned} \tag{7}$$

Using this notation the transfer matrix obeys

$$\mathbf{T}|0\rangle = |0\rangle\,, \qquad \langle 1|\mathbf{T} = \langle 1|\,, \tag{8}$$

i.e. the absorbing state is a ground state of the DK model and the transfer matrix conserves probability. Let us now select an arbitrary site j and define local operators $\mathbf{S}_0$, $\mathbf{P}_0$, and $\mathbf{W}$ by

$$\begin{aligned} \langle\sigma'|\mathbf{S}_0|\sigma\rangle &= \delta_{\sigma'_j,0} \prod_{i=1,i\neq j}^{L} \delta_{\sigma'_i,\sigma_i} \\ \langle\sigma'|\mathbf{P}_0|\sigma\rangle &= \delta_{\sigma'_j,0} \prod_{i=1}^{L} \delta_{\sigma'_i,\sigma_i} \\ \langle\sigma'|\mathbf{W}|\sigma\rangle &= \tau(0|\sigma_{j-1},\sigma_{j+1}) \prod_{i=1}^{L} \delta_{\sigma'_i,\sigma_i}\,. \end{aligned} \tag{9}$$

$\mathbf{S}_0$ turns site j into the inactive state, $\mathbf{P}_0$ projects onto states where site j is inactive, and $\mathbf{W}$ is a diagonal weight operator to be explained below. Notice that $\mathbf{S}_0$ conserves probability whereas $\mathbf{P}_0$ does not. Using these notations we can now express the local persistence probability $P_l(t)$ as

$$P_l(t) = \langle 1|(\mathbf{P}_0\mathbf{T})^t|i\rangle\,. \tag{10}$$

Here $|i\rangle$ denotes the initial probability distribution where the average over different initial configurations is taken into account. For example, the average over random initial conditions with particle density ρ corresponds to the initial distribution

$$|i\rangle = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_L} \prod_{i=1}^{L} \left[\rho\sigma_i + (1-\rho)(1-\sigma_i)\right] |\sigma\rangle\,. \tag{11}$$

In equation (10) the projector $\mathbf{P}_0$ removes all space-time histories which would turn site j into the active state giving just the local persistence probability $P_l(t)$. Notice that this expression is already properly normalized.

Now consider a DP process with an absorbing boundary inserted at site j. This process is described by a different transfer matrix $\tilde{\mathbf{T}}$ where site j is forced to be inactive:

$$\tilde{\mathbf{T}} = \mathbf{S}_0\mathbf{T}\,. \tag{12}$$

One can easily show that $\tilde{\mathbf{T}}$ is related to the full transfer matrix $\mathbf{T}$ by $\mathbf{P}_0\mathbf{T} = \tilde{\mathbf{T}}\mathbf{W}$ which implies that

$$P_l(t) = \langle 1|(\tilde{\mathbf{T}}\mathbf{W})^t|i\rangle\,. \tag{13}$$

Therefore the local persistence probability $P_l(t)$ is *exactly* equal to the expectation value of the diagonal operator $\mathbf{W}$ measured before each update in a DP process with an absorbing boundary. Equation (13) can be written as

$$P_l(t) = \left\langle \prod_{t'=0}^{t-1} \tau(0|\sigma_{j-1}(t'),\sigma_{j+1}(t')) \right\rangle\,, \tag{14}$$

where $\langle ...\rangle$ denotes the average over many independent realizations of a DP process with an absorbing boundary at site j, combined with an independent average over initial configurations according to the probability distribution $|i\rangle$. Note that $\tau(0|\sigma_{j-1}(t'),\sigma_{j+1}(t'))$ refers to the transition probability (1) of the DK model without dry wall at site j.

Because of the dry wall at site j, $\sigma_{j-1}(t)$ and $\sigma_{j+1}(t)$ are uncorrelated in an infinite system. Assuming that events where $\sigma_{j-1}(t) = \sigma_{j+1}(t) = 1$ are rare we may therefore approximate equation (14) by

$$P_l(t) \simeq \left\langle \exp\left[-\gamma \sum_{t'=0}^{t-1} \sigma_{j+1}(t')\right] \right\rangle\,, \tag{15}$$

where $\gamma = -2\log(1-p_1)$. Thus $P_l(t)$ is related to the integrated activity next to the boundary. As mentioned before, the surface activity of a (1+1)-dimensional DP process with an absorbing boundary has been carefully analyzed by series expansions and MC simulations in references [16,17]. It was observed that the activity next to the boundary decays according to a power law

$$\langle \sigma_{j\pm1}(t)\rangle \sim t^{-\beta'/\nu_{||}}\,. \tag{16}$$

Furthermore, it has been conjectured that $\beta' = \nu_{||} - 1$ which means that

$$\left\langle \sum_{t'=0}^{t-1} \sigma_{j+1}(t') \right\rangle \sim t^{1/\nu_{||}}\,. \tag{17}$$

If the average $\langle\ \rangle$ in equation (15) commuted with the exponential function, equation (17) would imply that $P_l(t)$ decays asymptotically as a *stretched exponential*

$$P_l(t) \sim \exp(-\gamma\, t^{1/\nu_{||}}) \tag{18}$$

rather than a power law. However, our numerical results strongly suggest that $P_l(t)$ does indeed decay as a power law (see Sect. 3.3). In fact, a stretched exponential of the form (18) is in obvious contradiction with the simulation data. Thus the average operation $\langle\ \rangle$ does certainly not commute with the exponential function. Nevertheless we are left with a puzzle: for both equations (15, 17) to decay algebraically, a delicate mechanism in the exponential function is needed, *i.e.* higher cumulants of the integrated surface activity have to match in a very specific way.

5 Relation to a directed percolation process with an active source

We will now show that the local persistence probability $P_l(t)$ can be related to a return probability in a (1+1)-dimensional DP process with a pointlike active source. More precisely, we will show that $P_l(t)$ is equal to the probability for a DP process with an active source to return to a state where all sites except for the source are inactive. We will prove this relation exactly in the case of directed bond percolation.

The mapping can qualitatively be understood as follows: consider a particular realization of a DP process (a single MC run) starting from initial conditions where all sites except for site j are active. Let us assume that in this particular realization site j has never become active up to time t_1. Such a realization is shown schematically in Figure 6. Obviously there are no open paths from the horizontal line $t = 0$ (line A in Fig. 6) to the vertical line at site j given by $0 < \tilde{t} \leq t_1$ (line B). Conversely, there is no open path *backwards* in time from line B to line A. In the special case of directed bond percolation we may now consider a DP process in reverse time direction using the same realization of open and closed bonds. Furthermore, let us assume that we force site j to be active along line B, *i.e.* we impose an active source at this location. If there is no open path from B to A, activity will not percolate from

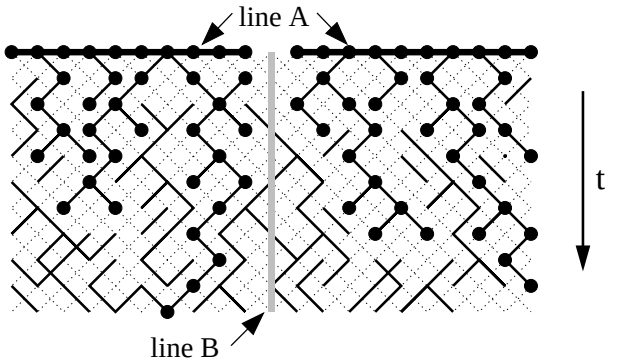

Fig. 6. Mapping of the local persistence problem onto a directed bond percolation process with a steady source in the center. The figure shows a particular realization of open bonds (solid lines) on a tilted square lattice (dotted lines) in directed bond percolation. Active sites are marked by bold dots. Initially all sites are in the active state (line A). Activity then percolates through the system without touching site j (line B). In reverse time direction this means that there is no open path backwards from line B to line A.

line B to line A in the "time-reversed" process. In other words, the reverse process returns to its initial condition where all sites except for the source are inactive.

We now prove this mapping for the special case of directed bond percolation. More precisely we show that in this case the probability $R(t)$ of a DP process with an active source to return to its initial condition where all sites except for site j are inactive is *exactly* equal to the persistence probability $P_l(t)$. Using the notations of Section 4, let us introduce two further operators $\mathbf{S}_1$ and $\mathbf{U}$:

$$\langle\sigma'|\mathbf{S}_1|\sigma\rangle = \delta_{\sigma'_j,1} \prod_{i=1,i\neq j}^{L} \delta_{\sigma'_i,\sigma_i}$$

$$\langle\sigma'|\mathbf{U}|\sigma\rangle = \prod_{i=1}^{L}(1-\delta_{\sigma'_i,1}\delta_{\sigma_i,1})\,. \tag{19}$$

The operator $\mathbf{S}_1$ turns site j into the active state whereas the operator $\mathbf{U}$ is a symmetric transformation matrix. One can easily verify that for $p_2 = p_1(2-p_1)$, *i.e.*, for directed bond percolation, the following relation holds:

$$\mathbf{UT} = \mathbf{T}^T\mathbf{U}\,. \tag{20}$$

Furthermore we have

$$\mathbf{UP}_0 = \mathbf{S}_1^T\mathbf{U}\,, \qquad \mathbf{US}_0 = \mathbf{S}_1^T\mathbf{U}\,. \tag{21}$$

By commuting the matrix $\mathbf{U}$ to the right and transposing the resulting expression, we can now rewrite the local persistence probability $P_l(t)$ in equation (13) as

$$\begin{aligned} P_l(t) &= \langle 1|(\mathbf{P}_0\mathbf{T})^t|i\rangle \\ &= \langle 0|\mathbf{U}(\mathbf{P}_0\mathbf{T})^t\mathbf{S}_0|a\rangle \\ &= \langle 0|(\mathbf{S}_1^T\mathbf{T}^T)^t\mathbf{S}_0^T\mathbf{U}|a\rangle \\ &= \langle a|\mathbf{US}_0(\mathbf{TS}_1)^t|0\rangle \\ &= \langle 0|\mathbf{S}_0(\mathbf{TS}_1)^t|0\rangle = R(t)\,, \end{aligned} \tag{22}$$

where we assumed the initial condition $|i\rangle = \mathbf{S}_0|a\rangle$ for the persistence measurement in which all sites except for site j are active. We also used the relations $\langle 1| = \langle 0|\mathbf{U}$ and $\langle a|\mathbf{U} = \langle 0|$. The resulting expression $\langle 0|\mathbf{S}_0(\mathbf{TS}_1)^t|0\rangle = R(t)$ is precisely the return probability of a DP process to its initial condition with an active source at site j, which completes the proof. Notice that similar arguments were used in reference [4] in order to derive the persistence exponent in the Glauber model.

Although this proof holds only for the case of directed bond percolation, the relation seems to be more general. In fact, we verified numerically that $P_l(t)$ and $R(t)$ exhibit the same power law behavior at various transition points in the DK model.

6 Conclusions

In the present work the problem of local and global persistence in directed percolation has been studied numerically. The results suggest that both the local and global persistence probabilities exhibit an algebraic decay in time at the critical point. The critical exponent θ_l turns out to be universal and independent of the initial density. The universality appears to be even stronger than in the Ising model where an explicit dependence on the initial density is found for θ_l. The reason might be an exponentially fast decaying memory of initial conditions in DP since active sites can spontaneously become inactive. We carefully analyzed our data in order to rule out any stretched exponential decay of the persistence probability. The measurement of the global persistence probability indicates that $\theta_l \geq \theta_g$ which contrasts with previously known cases where it was found that $\theta_g < \theta_l$. In order to consider the problem in a more general context, we related the local persistence probability to certain observables in a DP process with a dry and a wet wall, respectively. Introducing a transfer matrix formalism these relations were proven exactly for particular realizations of DP.

Various questions remain open. Recently it was shown by field-theoretical renormalization group analysis [14] that the global persistence exponent θ_g is indeed independent from other DP exponents. Whether the same is true for the local persistence exponent θ_l is not yet clear. The numerical value $\theta_l \approx 1.50(1)$ suggests that the exact value could be $3/2$, a possibility which cannot be ruled out as an integer exponent was also observed in a DP process with an absorbing boundary [16,17]. Finally, it would be interesting to investigate the same problem in higher dimensions.

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